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A new perspective on realizability of turbulence models

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In the second-moment-closure (SMC) method of turbulence modelling, measures to ensure a realizable turbulence model are currently limited to constraining the Reynolds stress to physically plausible values. These constraints address neither the realizability of the other statistical moments (e.g. pressure–strain correlation) nor the underlying causes of unrealizable Reynolds stress. For achieving increased consistency with flow physics in SMC, we propose the additional requirement that the closure model for each of the unclosed statistical moments in the Reynolds stress equation be individually realizable. We then proceed to derive two realizability constraints on the rapid-pressure statistics: (i) the rapid pressure-gradient variance must be positive which leads to the requirement that the M_{ijkl} tensor must be positive semi-definite, and (ii) the rapid pressure–strain correlation closure must satisfy the Schwarz inequality. Calculations with currently popular models show that unrealizable rapid-pressure–strain correlation precedes unrealizable Reynolds stress. It is also demonstrated that when the Launder, Reece and Rodi (LRR) rapid-pressure–strain correlation model is modified (truncated) to satisfy the new constraints, Reynolds stress realizability is always preserved. These findings clearly indicate that an unrealizable closure model is the cause of Reynolds stress realizability violation and highlight the importance of the new constraints.

1. Introduction

The realizability requirement enunciates the rudimentary expectation that an acceptable turbulence closure expression be based on the statistics of a velocity field that is physically achievable or realizable. The covariance tensor, or the Reynolds stress, of any velocity field governed by the Navier–Stokes equations is positive semi-definite exhibiting two important characteristics: the diagonal components (energies) are non-negative and the off-diagonal components satisfy the Schwarz inequality. For second-moment closures (SMC), the realizability constraint as proposed by Schumann (1977) requires that a turbulence model yield Reynolds stresses that satisfy these conditions (see also Lumley 1978). Over the last two decades, the Schumann realizability constraint has served as the theoretical basis for several turbulence models (e.g. Johansson & Hallback 1994; Ristorcelli, Lumley & Abid 1995; Sjögren & Johansson 2000). Realizability is also of great practical importance, for it can lead to computationally more robust models (Sjögren & Johansson 2000). In this paper, we revisit the realizability issue and present a new perspective.

We suggest that realizability is a broader issue than requiring that the Reynolds stress be a positive semi-definite tensor. While realizable Reynolds stress is a necessary condition for a physically plausible model, it is not sufficient. We propose that the physical accuracy of a turbulence model can be improved by requiring that all statistical moments directly modelled with closure expressions (e.g. pressure–strain correlation) and those implied by the closures must all be realizable. While the theoretical justification for this extended or comprehensive realizability condition is to achieve improved consistency with turbulence physics, there is also an important practical consideration. In second-moment closure (SMC) methodology, Reynolds stress is calculated from an evolution equation. The evolution rate is composed of contributions from production, pressure–strain correlation, turbulent transport, dissipation and other processes. Most of these terms require closure modelling. It is logical then that the most probable cause of unrealizable calculated stress is unphysical models for the individual terms contributing towards the evolution rate. One of the most fundamental prerequisites of a physically acceptable closure model is that it be realizable. Each unclosed SMC term is a statistical moment, the closure model for which must satisfy certain realizability constraints. For example, an unrealizable pressure–strain correlation model is one in which the closure expression cannot possibly be obtained from any physically permissible (real) pressure and velocity fields. Closure models not satisfying the appropriate realizability constraints (given in § 2) must be considered unphysical and unacceptable.

The traditional approaches attempt to enforce realizability of Reynolds stress without due consideration of the realizability of the underlying closure models. Although such a method may lead to apparently plausible Reynolds stresses, the dynamics of such SMC models will certainly be inconsistent with turbulence physics. Such realizability enforcement will defeat the ultimate goal which is to develop turbulence models with a high degree of fidelity to the Navier–Stokes equation.

In this paper, we will focus on the realizability issues of the rapid pressure–strain correlation term. Exploiting the divergence-free character of an incompressible flow field, we derive realizability constraints on the rapid pressure–strain correlation term. Simple analysis shows that many popular linear and nonlinear models do not satisfy these constraints. Detailed model calculations are performed in homogeneous turbulence to demonstrate unequivocally that the onset of unrealizable Reynolds stress is preceded by unrealizable rapid pressure–strain correlation. This establishes the cause–effect relationship between the unrealizable pressure–strain correlation model and the unrealizable Reynolds stress, highlighting the importance of the new constraints.

An introduction to the rapid pressure–strain correlation modelling is given in § 2. In § 3, we discuss the current realizability approach and motivate the need for additional constraints. The new realizability constraints on the rapid-pressure statistics are derived in § 4. Section 5 contains results and inferences. We close with a discussion in § 6.

2. Rapid pressure–strain correlation

The Reynolds stress evolution equation is given by:

$$\frac{\mathrm{D}\overline{u_i u_j}}{\mathrm{D}t} = P_{ij} - \phi_{ij} - \varepsilon_{ij} + T_{ij}, \tag{2.1}$$

where $\overline{u_i u_j}$ is the Reynolds stress. The various terms in (2.1) are the time-rate-of-change following the mean flow, production, pressure–strain correlation, dissipation and transport. Several of these terms require closure modelling.

Pressure–strain correlation modelling methodology for incompressible flows is derived from the early work of Chou (1945). We start with the Poisson equation for fluctuating pressure:

$$\frac{\partial^2 p}{\partial x_i \partial x_i} = -2(S_{ij}s_{ji} + W_{ij}w_{ji}) - (s_{ij}s_{ji} + w_{ij}w_{ji}). \tag{2.2}$$

Here, and throughout the remainder of the paper, the following notation will be used: S_{ij} and W_{ij} represent the strain and rotation rates of the mean flow; s_{ij} and w_{ij} represent those of the fluctuating field. Repeated italic indices will imply summation. The pressure fluctuations are composed of two parts:

$$p = p^r + p^s, \tag{2.3}$$

where the slow part (p^s) is the pressure due to the fluctuating velocity field:

$$\frac{\partial^2 p^s}{\partial x_i \partial x_i} = -(s_{ij}s_{ji} + w_{ij}w_{ji}). \tag{2.4}$$

The rapid pressure (p^r) arises from the interaction between the mean and fluctuating velocity fields. The Poisson equation for fluctuating rapid pressure can be written as

$$\frac{\partial^2 p^r}{\partial x_i \partial x_i} = -2(S_{ij}s_{ji} + W_{ij}w_{ji}). \tag{2.5}$$

Correspondingly, it is normal practice to decompose the pressure–strain correlation into slow and rapid parts:

$$\phi_{ij} = \phi^s_{ij} + \phi^r_{ij} \tag{2.6}$$

and model each component separately (Launder, Reece & Rodi 1975).

Using the Green's function solution to the Poisson equation, it can be shown that the (rapid) pressure velocity-gradient correlation tensor can be written in the form

$$\overline{p^r \frac{\partial u_i}{\partial x_j}} \equiv 2\frac{\partial \overline{U_k}}{\partial x_l} M_{iljk}. \tag{2.7}$$

The fourth-order tensor M_{iljk} is given by

$$M_{iljk} = -\frac{1}{4\pi}\int \frac{1}{|\boldsymbol{r}|}\frac{\partial^2 R_{il}}{\partial r_j \partial r_k}\,\mathrm{d}\boldsymbol{r} \tag{2.8}$$

where R_{il} is the two-point velocity correlation: $R_{il}(\boldsymbol{r}) \equiv \overline{u_i(\boldsymbol{x})u_l(\boldsymbol{x}+\boldsymbol{r})}$. It is then straightforward to show that the rapid pressure–strain correlation can be expressed as

$$\phi^r_{ij} = 2\overline{p^r s_{ij}} \equiv 2\frac{\partial \overline{U_k}}{\partial x_l}(M_{iljk} + M_{jlik}). \tag{2.9}$$

Modelling rapid pressure–strain correlation is tantamount to modelling the M_{ijkl} tensor. Complete description of the M_{ijkl} tensor requires knowledge of the Reynolds stress (componentiality tensor) and the so-called dimensionality tensor (D_{ij}) as is explained in Kassinos, Reynolds & Rogers (2001). The evolution of the dimensionality tensor is governed, to a large extent, by the initial dimensionality and the mean-velocity gradients. The dimensionality tensor is unclosed in terms of the single-point tensors used in the conventional SMC methods. For a given Reynolds stress, the rapid pressure–strain correlation can take a range of values depending on the dimensionality of the turbulence. This is the crux of the closure problem. While carrying evolution equations for dimensionality is an option, it is not generally undertaken because of the added computational expense. The practice in traditional SMC modelling is to postulate a closure expression for the M_{ijkl} tensor that is based on some implied average value of the dimensionality tensor. In effect, the conditional (based on a specific value of the dimensionality tensor) correlation between pressure and strain fluctuations is modeled with the unconditional correlation (based on some average dimensionality tensor). Thus, the need for the dimensionality tensor is obviated, at the expense of accuracy.

Lacking knowledge of the dimensionality tensor, the closure expression for pressure–strain correlation is generally required to satisfy several physical and mathematical constraints to ensure some degree of fidelity to the flow physics. A complete list of currently mandated constraints can be found in Pope (2000). Three of the important constraints are that a rapid pressure–strain correlation model must (i) be linear in Reynolds stress; (ii) conform to a specific functional form (given in (2.9)); and, (iii) lead to a realizable Reynolds stress. While the need for the realizability requirement is now clear, the linearity requirement also stems from important turbulence physics. It is easy to see that the Poisson equation for rapid pressure is linear in fluctuating velocity (Reynolds 1976). Therefore, the rapid or linear pressure term cannot increase the number of (Fourier) modes of fluctuation. In other words, the rapid pressure term conserves the number of fluctuating modes. Any nonlinear model cannot preserve this important physical characteristic of rapid pressure. The M_{ijkl}-form requirement is a consequence of the form of the Green's function solution of the Poisson equation. As pointed out in Pope (2000), currently, there exists no rapid pressure–strain correlation model that satisfies all of the required constraints. Specifically, none of the current models satisfy both realizability and linearity requirements. In fact, it has been shown that it is impossible to satisfy realizability fully with linear models (Lumley 1978). While realizability is an important constraint, it is equally desirable to achieve it in a manner as closely consistent with other turbulence physics (such as the linearity requirement) as possible. In this paper, we propose additional constraints on ϕ'_{ij} and M_{ijkl} that could be expected to aid future development of such closure models.

3. Realizability constraints

We will first discuss the strong and weak forms of Schumann constraints. Realizability of Reynolds stresses is most conveniently considered in terms of the determinant of the normalized Reynolds-stress tensor:

$$F \equiv \det\left(\frac{\overline{u_i u_j}}{\frac{1}{3}\overline{u_k u_k}}\right). \tag{3.1}$$

In the realizable state-space of Reynolds stress (Lumley invariant triangle), F is positive: F is negative through most of the unrealizable space. The realizable and the unrealizable Reynolds stress state-spaces are separated by two-componential turbulence in which the determinant F vanishes. The current realizability conditions constrain the behaviour of the model in the neighbourhood of $F = 0$.

Strong realizability

The strong Reynolds stress realizability condition requires the turbulence model to satisfy the following conditions (Pope 2000):

$$\left(\frac{\mathrm{d}F}{\mathrm{d}T}\right)_{F=0} = 0, \quad \left(\frac{\mathrm{d}^2F}{\mathrm{d}^2T}\right)_{F=0} > 0. \tag{3.2}$$

The strong realizability condition permits accessibility to two-component turbulence. Rigorous implementation of strong realizability constraints to develop closure models is best exemplified in Ristorcelli, Lumley & Abid (1995), Sjögren & Johansson (2000), etc. In fact, in Johansson & Hallback (1994), a recipe for deriving a rapid pressure–strain correlation model that satisfies the strong realizability constraint at any order of tensor expansion is provided. It was found that the fourth-order model was successful over a wider range of flows than typical linear models. In Sjögren & Johansson (2000), the use of a strong realizability constraint to develop slow pressure-correlation and anisotropic dissipation models are also summarized.

Weak realizability

Many authors point to major difficulties in enforcing the strong realizability constraint and opt for the easier to implement weak realizability version. The weak realizability constraint simply requires that the rate of change of F be positive as the two-componential limit is approached (Pope 2000):

$$\left(\frac{\mathrm{d}F}{\mathrm{d}T}\right)_{F=0} > 0. \tag{3.3}$$

This constraint does not permit access to two-component turbulence. The stochastic realizability analysis of Durbin & Speziale (1994) is a variant of the weak-realizability approach. They demonstrate that every turbulence model equation can be represented by an equivalent stochastic equation. The requirement that the coefficient of the random forcing term of the stochastic equation be real yields constraints on the pressure–strain correlation model. This approach leads to the dependence of the rapid pressure–strain correlation model coefficients on the coefficients in the dissipation and the slow pressure–strain correlation models. Such interdependence of model coefficients is not physically justifiable. For example, if a turbulence model yields unrealizable results in the rapid distortion limit, it is the rapid pressure–strain correlation model that requires modification. Slow pressure–strain correlation and dissipation processes are completely irrelevant in this limit and modifying their models to achieve realizability must be deemed unphysical. The realizability issues of each process should be considered independently, as is demonstrated in Sjögren & Johansson (2000).

Need for further constraints

In the second-moment closure approach, Reynolds stress is calculated using a modelled evolution equation. Any non-physical behaviour of the Reynolds stress must originate from unphysical closure models in its evolution equation. In Navier–Stokes physics, the realizability of the Reynolds stress is achieved while maintaining all the contributing statistical moments individually realizable. Any realizability analysis is incomplete without the examination of the realizability of the individual closure models that contribute towards Reynolds stress evolution. The strong realizability approach indirectly addresses this issue, but only in the two-componential turbulence limit. The weak realizability approach, as currently practised, completely ignores the realizability of individual moments in the Reynolds stress evolution equation. Thus, both the strong and the weak realizability approaches can and do (as will be demonstrated later) permit unrealizable pressure–strain correlation values.

In order to achieve a greater degree of consistency with turbulence physics, we recommend that the closure model for each moment in the Reynolds stress evolution equation be individually realizable, not only in the two-componential limit, but also at all other states of turbulence. Physically achievable or realizable models for each of the unclosed statistical moments must satisfy the following requirements:

(i) Auto-covariances must be non-negative.

(ii) Cross-covariances must satisfy the Schwarz inequality.

In this paper, we address the realizability of the rapid pressure–strain correlation term. In most modelling methods, it is the M_{ijkl} tensor that is first modelled – see for example, Launder, Reece & Rodi (1975); Ristorcelli, Lumley & Abid (1995) or equation (54) in Sjögren & Johansson (2000) – and the pressure–strain correlation closure is simply obtained from (2.9). Hence, postulating constraints on M_{ijkl} is as effective and useful as constraining pressure–strain correlation itself. Here, we derive constraints on the M_{ijkl} tensor that guarantee realizable rapid pressure–strain correlation.

The need for further constraints on M_{ijkl} can also be motivated from the turbulence structure tensors point of view. As Kassinos *et al.* (2001) point out, a complete one-point statistical description of turbulence requires specification of several tensors: componentiality or Reynolds stress (R_{ij}); dimensionality (D_{ij}); circulicity (F_{ij}); inhomogeneity (C_{ij}); and, finally, stropholysis (Q_{ijk}). In general, each tensor carries independent information and all of them are required in order to fully specify the one-point statistical state of turbulence. It then stands to reason that closures for all the tensors should be individually realizable. In the special case of incompressible homogeneous turbulence, these structure tensors can be related to the M_{ijkl} tensor (Kassinos *et al.* 2001):

$$Q_{ijk} = \epsilon_{ipq} M_{jqpk},\ R_{ij} = M_{ijrr},\ D_{ij} = M_{rrij},\ F_{ij} = \epsilon_{imp}\epsilon_{jrs} M_{psrm}, \tag{3.4}$$

where ϵ_{ijk} is the alternating tensor. Therefore, in homogeneous incompressible turbulence, all physical and realizability issues are embodied in M_{ijkl} tensor. Thus the realizability of the M_{ijkl} tensor is much more fundamental than that of Reynolds stress alone. Hence, constraints that can guarantee the physical fidelity of this tensor are of great importance.

4. Realizability of rapid pressure moments

In homogeneous turbulence, the pressure–strain correlation (ϕ_{ij}) is identical in magnitude and opposite in sign to the velocity pressure-gradient correlation (Π_{ij}):

$$\Pi_{ij} = \overline{u_i \frac{\partial p}{\partial x_j} + u_j \frac{\partial p}{\partial x_i}} = -2\overline{p s_{ij}} = -\phi_{ij}. \tag{4.1}$$

The velocity pressure-gradient correlation is more fundamental as it is the one that appears in the Reynolds stress closure equation before decomposition into homogeneous and inhomogeneous components. We will now investigate the realizability bounds on the rapid portion of the velocity (rapid) pressure-gradient correlation (Π'_{ij}):

$$\Pi'_{ij} = \overline{u_i \frac{\partial p'}{\partial x_j} + u_j \frac{\partial p'}{\partial x_i}} = -2\overline{p' s_{ij}} = -\phi'_{ij}. \tag{4.2}$$

The magnitude of this correlation is bounded by

$$|\Pi'_{\alpha\beta}| = \left| \overline{u_\alpha \frac{\partial p'}{\partial x_\beta}} + \overline{u_\beta \frac{\partial p'}{\partial x_\alpha}} \right| \leqslant \left| \overline{u_\alpha \frac{\partial p'}{\partial x_\beta}} \right| + \left| \overline{u_\beta \frac{\partial p'}{\partial x_\alpha}} \right|. \tag{4.3}$$

The Schwarz inequality as applied to the last two correlations on the extreme right requires

$$\left| \overline{u_\alpha \frac{\partial p'}{\partial x_\beta}} \right| \leqslant \overline{u_\alpha u_\alpha}^{1/2} \; \overline{\frac{\partial p'}{\partial x_\beta} \frac{\partial p'}{\partial x_\beta}}^{1/2}. \tag{4.4}$$

(Whereas repeated italic indices imply summation, Greek indices do not.) As a consequence, the rapid portion of velocity pressure-gradient correlation must be bounded by:

$$|\Pi'_{\alpha\beta}| \leqslant \overline{u_\alpha u_\alpha}^{1/2} \; \overline{\frac{\partial p'}{\partial x_\beta} \frac{\partial p'}{\partial x_\beta}}^{1/2} + \overline{u_\beta u_\beta}^{1/2} \; \overline{\frac{\partial p'}{\partial x_\alpha} \frac{\partial p'}{\partial x_\alpha}}^{1/2}. \tag{4.5}$$

Since $\overline{(\partial p'/\partial x_\beta)(\partial p'/\partial x_\beta)} \leqslant \overline{(\partial p'/\partial x_i)(\partial p'/\partial x_i)}$, the following inequality must hold:

$$|\Pi'_{\alpha\beta}| \leqslant \left(\overline{u_\alpha u_\alpha}^{1/2} + \overline{u_\beta u_\beta}^{1/2}\right) \overline{\frac{\partial p'}{\partial x_i} \frac{\partial p'}{\partial x_i}}^{1/2} \equiv G_{\alpha\beta}, \tag{4.6}$$

where the second identity defines the tensor $G_{\alpha\beta}$. Because we replace $\overline{(\partial p'/\partial x_\beta)(\partial p'/\partial x_\beta)}$ in the inequality by the larger $\overline{(\partial p'/\partial x_i)(\partial p'/\partial x_i)}$, we must regard the constraint in (4.6) as less stringent than required to satisfy the Schwarz inequality. This inequality is valid for homogeneous and inhomogeneous flows. From (4.6), it can be seen that the realizability bounds on the rapid pressure–strain correlation depend on the componentiality (via $\overline{u_\alpha u_\alpha}$) and dimensionality (via $\overline{(\partial p'/\partial x_\beta)(\partial p'/\partial x_\beta)}$) of the turbulence.

The inequality in (4.6) forms the basis of the new constraints developed in this paper. In the limit of two-componential turbulence, this inequality yields the strong realizability constraint: if velocity fluctuations in the α-direction vanish, then

$$\overline{u_\alpha u_\alpha} = 0, \tag{4.7}$$

which immediately implies

$$|\phi'_{\alpha\alpha}| = |\Pi'_{\alpha\alpha}| \leqslant \overline{u_\alpha u_\alpha}^{1/2} \; \overline{\frac{\partial p'}{\partial x_\beta} \frac{\partial p'}{\partial x_\beta}}^{1/2} = 0. \tag{4.8}$$

Apart from this requirement, the inequality in (4.6) has not been exploited for formulating all other possible modelling constraints. For example, it is obvious that when the pressure-gradient variance vanishes, so must the pressure–strain correlation. Yet, this fundamental requirement has never been posed as a modelling constraint. This is probably because rapid pressure-gradient statistics (indicative of the dimensionality of turbulence) are not considered in traditional modelling approaches where only the Reynolds stresses (componentiality of turbulence) are solved for explicitly. In this work, we will derive an exact analytical expression for rapid pressure-gradient variance. Knowledge of rapid pressure-gradient variance opens up the possibility of a new set of constraints on the rapid pressure–strain correlation that are much more general than the strong realizability requirement.

Rapid pressure-gradient variance

Multiplying (2.5) through by the rapid pressure and taking the average leads to:

$$\overline{p'\frac{\partial^2 p'}{\partial x_i \partial x_i}} = -2S_{ij}\overline{p's_{ij}} - 2W_{ij}\overline{p'w_{ji}}. \tag{4.9}$$

In homogeneous turbulence which is of interest here, we can further write

$$\overline{p'\frac{\partial^2 p'}{\partial x_i \partial x_i}} = -\overline{\frac{\partial p'}{\partial x_i}\frac{\partial p'}{\partial x_i}} = -2S_{ij}\overline{p's_{ij}} - 2W_{ij}\overline{p'w_{ji}} = -2\frac{\partial \overline{U}_i}{\partial x_j}\overline{p'\frac{\partial u_j}{\partial x_i}}. \tag{4.10}$$

It was shown in the previous section that the rapid pressure–strain correlation tensor is related to the fourth-order tensor M_{ijkl}. The relationship between this tensor and the pressure-gradient variance can be easily derived from (2.7) and (4.10):

$$\overline{\frac{\partial p'}{\partial x_i}\frac{\partial p'}{\partial x_i}} = 4\frac{\partial \overline{U}_j}{\partial x_i}\frac{\partial \overline{U}_k}{\partial x_l}M_{iljk}. \tag{4.11}$$

Thus, the rapid pressure-gradient variance can be obtained as a function of the rapid pressure–strain correlation. In purely strained flows, the relationship is even simpler:

$$\overline{\frac{\partial p'}{\partial x_i}\frac{\partial p'}{\partial x_i}} = 2S_{ij}\overline{p's_{ij}}. \tag{4.12}$$

It has to be pointed out that Kassinos *et al.* (2001) derive an expression for pressure-gradient variance in the rapid reference-frame rotation limit. The expression involves the rotation rate and the circulicity tensor which contains information about the large-scale vorticity. The circulicity tensor is not known in the traditional second-moment closure modelling.

4.1. *New realizability constraints*

With the knowledge of rapid pressure-gradient variance, we will now formulate the new realizability constraints in one-, two- and three-componential homogeneous turbulence. These pressure–strain correlation constraints account for dimensionality as well as componentiality, whereas, the Schumann constraint takes only componentiality into consideration.

Constraint 1. In two-componential turbulence (with no velocity fluctuations in the α direction), the realizability of rapid pressure–strain correlation requires

$$\phi'_{\alpha\alpha} = \Pi'_{\alpha\alpha} = 0. \tag{4.13}$$

The values of the off-diagonal terms will depend on the dimensionality of the turbulence. This is similar to the strong realizability requirement. Alternatively, the realizability constraint on M_{ijkl} in the two-component limit is

$$M_{\alpha\alpha ij} = 0. \tag{4.14}$$

In one-componential turbulence, all three diagonal pressure–strain correlation components must be zero. The diagonal components in the direction of zero fluctuating velocity clearly must be zero for the same reason as above. The correlation component in the third direction is zero for a different reason. Continuity requires that the fluctuating pressure gradient be zero in the direction of velocity. Thus, in one-dimensional turbulence (if β is the direction of non-zero velocity) the realizability constraints are:

$$\left.\begin{aligned} &\text{for } \alpha \neq \beta: \quad \overline{u_\alpha u_\alpha} = 0, \text{ implying } \phi'_{\alpha\alpha} = 0;\ M_{\alpha\alpha ij} = 0, \\ &\text{for } \alpha = \beta: \quad \overline{\frac{\partial p'}{\partial x_\alpha}\frac{\partial p'}{\partial x_\alpha}} = 0, \text{ implying } \phi'_{\alpha\alpha} = 0;\ M_{ij\alpha\alpha} = 0. \end{aligned}\right\} \tag{4.15}$$

The pressure–strain correlation constraints are valid for the slow term as well.

Constraint 2. The rapid pressure–strain correlation model must lead to non-negative values for the rapid pressure-gradient variance. Physically, this ensures that the pressure fluctuations are real. Thus, the first of two realizability constraints on M_{ijkl} is

$$\frac{\partial \overline{U}_j}{\partial x_i}\frac{\partial \overline{U}_k}{\partial x_l} M_{iljk} \geqslant 0. \tag{4.16}$$

This inequality should hold for any arbitrary non-zero mean-velocity gradient tensor. Hence, M_{ijkl} must be positive semi-definite. Unless M_{ijkl} is positive semi-definite, there will be a mean velocity field, for which the fluctuating rapid pressure gradient will be negative.

For purely strained flows, the positivity condition on the pressure-gradient variance can also be written as

$$S_{ij}\phi'_{ij} \geqslant 0. \tag{4.17}$$

Constraint 3. The pressure-gradient variance must not only be positive, but also be large enough to satisfy the Schwarz inequality on rapid pressure–strain correlation. This can be guaranteed if the model satisfies the following inequality (from (4.11) and (4.6)):

$$2|\overline{p's_{\alpha\beta}}| \equiv |\Pi'_{\alpha\beta}| \leqslant 2\left(\overline{u_\alpha u_\alpha}^{1/2} + \overline{u_\beta u_\beta}^{1/2}\right)\left[\frac{\partial \overline{U}_j}{\partial x_i}\frac{\partial \overline{U}_k}{\partial x_l} M_{iljk}\right]^{1/2}. \tag{4.18}$$

Satisfaction of this constraint in conjunction with constraint 2 will ensure realizable velocity pressure-gradient correlation. For purely strained flows, the above constraint can be expressed directly in terms of pressure–strain correlation:

$$\phi'_{ij} \equiv |\Pi'_{\alpha\beta}| \leqslant \left(\overline{u_\alpha u_\alpha}^{1/2} + \overline{u_\beta u_\beta}^{1/2}\right)[S_{ij}\phi'_{ij}]^{1/2}. \tag{4.19}$$

It can be recognized that the first constraint (equation (4.13)), derived for one- and two-componential turbulence, is equivalent to the strong Reynolds stress realizability constraint. However, constraints 2 (equation (4.16)) and 3 (equation (4.18)) are not addressed in existing literature. In light of these new constraints several questions arise:

(i) Do the current models violate the new realizability constraints? If not, the new constraints are only of academic interest.

(ii) Does Reynolds-stress realizability automatically imply realizability of pressure–strain correlation? If yes, then the current constraints would be redundant.

(iii) Conversely, does pressure–strain correlation realizability guarantee Reynolds-stress realizability? If yes, then the new realizability constraints are more complete and the Reynolds-stress realizability is redundant.

(iv) Or, are the old and new realizability constraints mutually exclusive so that both should be required of the pressure–strain correlation models?

In an attempt to find the answers, we perform model calculations in the next section.

5. Results, discussion and implications

We will now examine the realizability of the calculated Reynolds stress and modeled rapid pressure–strain correlation in SMC computations employing various models. First, we will focus our attention on linear models. Linear models not only satisfy the fundamental linearity requirement, but are more easily amenable to fully explicit algebraic Reynolds stress reduction (Girimaji 1996). The linear (and quasi-linear) rapid pressure–strain correlation models considered in this work can be represented in the form

$$\phi_{ij} = -\left(C_1^0\varepsilon + C_1^1 P\right)b_{ij} + C_2 K S_{ij} + C_3 K\left(b_{ik}S_{jk} + b_{jk}S_{ik} - \tfrac{2}{3}b_{mn}S_{mn}\delta_{ij}\right) + C_4 K(b_{ik}W_{jk} + b_{jk}W_{ik}), \quad (5.1)$$

where the C terms are model coefficients; $K = \overline{u_i u_i}/2$; $\varepsilon = \nu\overline{(\partial u_i/\partial x_j)(\partial u_i/\partial x_j)}$; and $b_{ij} = (\overline{u_i u_j}/2K) - \delta_{ij}/3$. The following models were tested: the isotropization of production (IP) model; the LRR model (Launder *et al.* 1975); quasi-linear and nonlinear versions of the SSG model (Speziale, Sarkar & Gatski 1992); the JM model (Jones & Musonge 1988); and the Girimaji model (2000). For the model coefficient values the reader is referred to the original works. The rapid portion of the various models can be written as

$$\phi'_{ij} = -C_1^1 P b_{ij} + C_2 K S_{ij} + C_3 K\left(b_{ik}S_{jk} + b_{jk}S_{ik} - \tfrac{2}{3}b_{mn}S_{mn}\delta_{ij}\right) + C_4 K(b_{ik}W_{jk} + b_{jk}W_{ik}). \quad (5.2)$$

Results from the IP, LRR and SSG models are now presented. It is known that these models may not satisfy strong or ‘individual’ realizability in one- or two-componential limits. We will first investigate the extent of violation in these limiting states of turbulence and consistency with RDT (rapid distortion theory) equations.

One-component turbulence limit

This regime of turbulence is of limited practical importance. While inaccuracies can be tolerated, an unrealizable closure expression at this limit must be regarded as an indication of serious flaws in the modelling methodology. At this limit, the constraint derived in the previous section requires that all the diagonal components of the model must be zero. For all α, we must have

$$\phi'_{\alpha\alpha} = 0 = -C_1^1 P b_{\alpha\alpha} + C_2 K S_{\alpha\alpha} + C_3 K\left(b_{\alpha k}S_{\alpha k} + b_{\alpha k}S_{\alpha k} - \tfrac{2}{3}b_{mn}S_{mn}\right) + C_4 K(b_{\alpha k}W_{\alpha k} + b_{\alpha k}W_{\alpha k}). \quad (5.3)$$

If the modelled pressure–strain correlation component is to be zero, then the model coefficients must be either zero or scalar functions of anisotropy, strain-rate and rotation-rate tensors. In most of the popular models, these coefficients are non-zero constants. Thus, the current models are not even qualitatively consistent with this realizability condition. To further understand the model limitations, we now present their behaviour at one-component limit in purely strained flows. The models are tested in plane-strain, axisymmetric expansion and contraction flows. The nomenclature $1C\alpha$ refers to a one-component velocity field in which only u_α is non-zero. The results are summarized in table 1. Under the column marked ‘Exact’ two items are presented: (i) the analytical constraint on the magnitude of the pressure–strain correlation component and (ii) whether RDT permits evolution from that state of turbulence. The letter E is used to indicate that RDT permits evolution and N indicates no

IC	Flow	$\Pi_{\alpha\alpha}$	Exact RDT		IP		LRR		SSG	
1C1	AC	11	N	0	EU	0.8	EU	0.98	EU	1.486
		22		0		−0.4		−0.49		−0.743
		33		0		−0.4		−0.49		−0.743
	AE	11	N	0	ER	−1.6	ER	−1.964	ER	−2.972
		22		0		0.8		0.982		1.486
		33		0		0.8		0.982		1.486
	PS	11	N	0	EU	0.8	EU	0.982	EU	1.486
		22		0		−0.4		−0.4		−0.469
		33		0		−0.4		−0.582		−1.017
1C2	AC	11	N	0	ER	0.2	ER	0.109	EU	−0.039
		22		0		−0.4		−0.491		−0.743
		33		0		0.2		0.382		0.782
	AE	11	N	0	EU	−0.4	EU	−0.218	EU	0.078
		22		0		0.8		0.982		1.486
		33		0		−0.4		−0.764		−1.564
	PS	11	N	0	ER	0.4	ER	−0.4	ER	0.469
		22		0		−0.8		−0.182		−1.486
		33		0		0.4		0.582		1.017
1C3	AC	11	N	0	ER	0.2	ER	0.109	EU	−0.039
		22		0		0.2		0.382		0.782
		33		0		−0.4		−0.491		−0.743
	AE	11	N	0	EU	−0.4	EU	−0.218	EU	−0.078
		22		0		−0.4		−0.764		−1.564
		33		0		0.8		−0.982		1.486
	PS	11	N	0	N	0	EU	−0.182	EU	−0.547
		22		0		0		0.182		0.547
		33		0		0		0		0

TABLE 1. Comparison between model and RDT in one-component turbulence.

evolution. These 'exact' results are compared against model calculation. For each model, the numerical value of the pressure–strain correlation component is listed. Whether a model leads to evolution (E) or no evolution (N) is also indicated. Also shown is whether the model evolution yields realizable (ER) or unrealizable (EU) Reynolds stresses. These calculations employ the rapid portion of the pressure–strain correlation model only apart from the production term. This enables a clean comparison against RDT in which the slow pressure–strain correlation and dissipation effects are excluded.

All the linear models violate the pressure–strain correlation bounds in the one-dimensional turbulence limit. While the RDT result calls for no evolution of anisotropy in all the cases considered, the models predict spurious evolution. This unphysical evolution may or may not yield realizable Reynolds stresses. Hence, apart from being unrealizable, the models are also inconsistent with RDT physics.

Two-componential turbulence limit

This regime of turbulence is very important, since it represents the boundary between realizable and unrealizable Reynolds stress. At this limit of turbulence we

IC	Flow	$\Pi_{\alpha\alpha}$	Exact RDT		IP		LRR		SSG	
$2C1$	AC	11	N	0	ER	0.2	ER	0.109	ER	0.226
	AE	11	N	0	EU	−0.4	EU	−0.218	EU	−0.452
	PS	11	E	0	ER	0.2	ER	0.109	EU	0.226
$2C2$	AC	22	E	0	EU	−0.1	EU	−0.055	EU	−0.113
	AE	22	E	0	ER	0.2	ER	0.109	ER	0.226
	PS	22	E	0	EU	−0.2	EU	−0.109	EU	−0.226
$2C3$	AC	33	E	0	EU	−0.1	EU	−0.0545	EU	−0.113
	AE	33	E	0	ER	0.2	ER	0.109	ER	0.226
	PS	33	E	0	ER	0	ER	0	ER	0

TABLE 2. Comparison between model and RDT in two-component turbulence.

have,

$$\overline{u_\alpha u_\alpha} = 0, \quad \text{implying} \quad b_{\alpha\alpha} = -\tfrac{1}{3}. \tag{5.4}$$

The corresponding modelled rapid pressure–strain correlation component must be zero:

$$\phi'_{\alpha\alpha} = 0 = \tfrac{1}{3}C_1^1 P + C_2 K S_{ij} + C_3 K \left(b_{\alpha k} S_{\alpha k} + b_{\alpha k} S_{\alpha k} - \tfrac{2}{3} b_{mn} S_{mn}\right) + C_4 K (b_{\alpha k} W_{\alpha k} + b_{\alpha k} W_{\alpha k}). \tag{5.5}$$

Any model with fixed coefficients is inherently incapable of satisfying the required constraints given in (4.13). Again, we look to compare the model against exact RDT for further insight. The results from model computations and RDT are presented in table 2. The notation $2C\alpha$ represents a two-componential velocity field in which $u_\alpha = 0$. The observations are very similar to those in one-componential turbulence.

The overall conclusion is that the current linear pressure–strain correlation models with constant coefficients do not satisfy the realizability requirement by a significant margin. This observation is consistent with that of Lumley (1978). Further, they are inconsistent with rapid distortion theory as they can yield spurious evolution of the anisotropy tensor when RDT indicates no evolution. Even when the model evolution is realizable, it can be inconsistent with RDT. The physical significance of these issues is discussed in more detail in Girimaji, Jeong & Poroseva (2003).

Three-componential turbulence

In this comparison, all terms including slow pressure–strain correlation and dissipation are employed in the model computation. Rapid ($SK_0/\varepsilon_0 = 50$), moderate ($SK_0/\varepsilon_0 = 15$) and mild ($SK_0/\varepsilon_0 = 5$) distortion rates are considered. Various homogeneous mean flow cases (homogeneous shear, plane strain, axisymmetric expansion and contraction) were investigated. Calculations were carried out for a variety of different initial Reynolds stress anisotropies. For each flow, over three hundred solution trajectories with the initial conditions systematically covering the entire Lumley triangle for all possible permutations of the anisotropies were investigated.

Realizability violations were observed with every linear model – for a sizable sub-set of initial conditions – in each flow considered. The detailed results are presented in Sambasivam, Girimaji & Poroseva (2004). The most important findings are: (i) in all

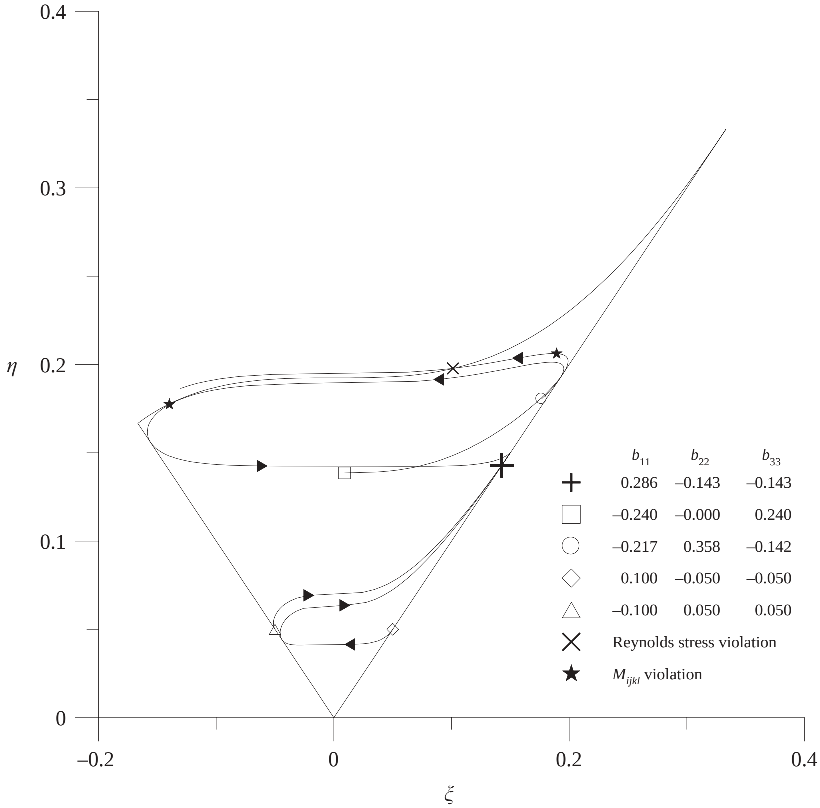

FIGURE 1. SSG model trajectories in homogeneous turbulence with axisymmetric-expansion mean flow. Anisotropy of the equilibrium point (+) and initial conditions are indicated. Locations of Reynolds stress and rapid pressure–strain realizability violations are marked ($SK_0/\varepsilon_0 = 5$).

observed transgressions, the pressure–strain correlation constraints (given in (4.18)) are violated before Reynolds stresses become unrealizable; (ii) the subset of initial conditions leading to unrealizable behaviour grows larger with increasing magnitude of mean-flow distortion; (iii) purely sheared flows exhibit smaller unrealizable regions than purely strained flows; and (iv) for a given mean flow, the size of the unrealizable region depends on the closure model used. For example, the LRR model generally exhibits smaller unrealizable regions than the SSG model. Here, we present some typical results from plane-strain and axisymmetric expansion flow calculations.

Axisymmetric expansion. We present results in the form of solution trajectories in the Lumley triangle (figure 1). All models considered exhibit only one attracting fixed point (equilibrium point) for realizable trajectories. When initial conditions are nearly isotropic, all models yield realizable trajectories that ultimately asymptote to the (structural) equilibrium point. Two such trajectories (calculated using the SSG model) are shown in figure 1 with initial conditions marked by $\diamond$ and $\triangle$.

When initial anisotropies are large, unrealizable Reynolds stress trajectories are encountered. Each trajectory that ultimately yields unrealizable Reynolds stresses first violates the new pressure–strain correlation constraint (equation (4.18)). One such (SSG) model trajectory in shown in the figure with its initial condition denoted by $\bigcirc$.

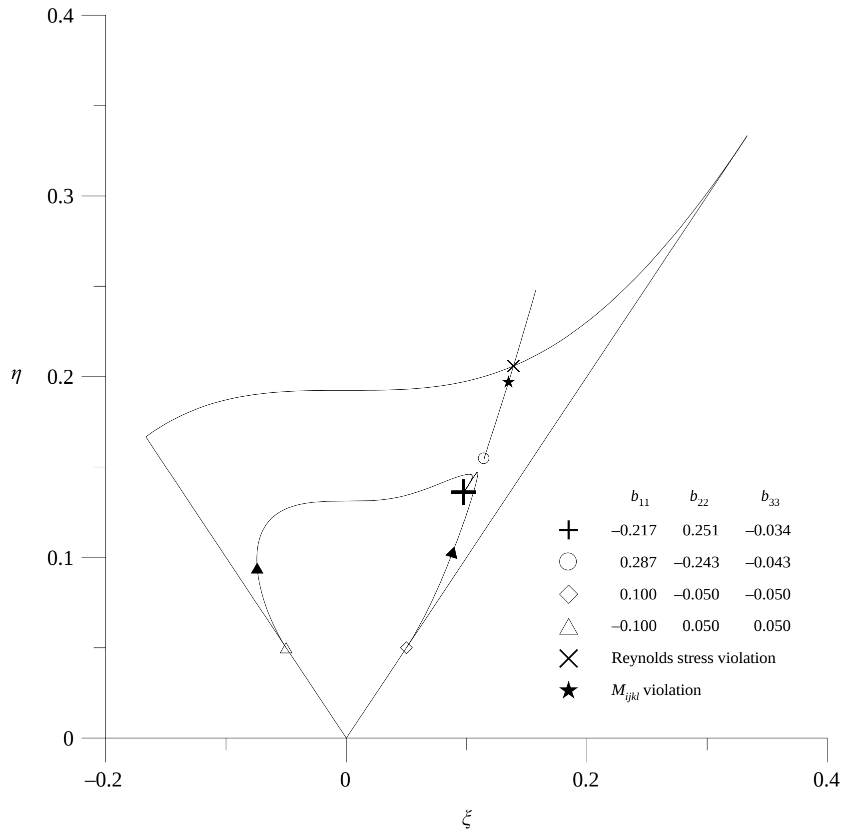

FIGURE 2. LRR model trajectories in homogeneous turbulence with plain-strain mean flow Legend same as in figure 1.

With time, one of the energy components tends towards zero and ultimately becomes negative at the location marked by $\times$ at the upper boundary of the Lumley triangle. Before the two-componential limit is reached, the new pressure–strain correlation constraint is violated at the point marked with $\star$ in the figure.

For intermediate values of initial anisotropy, the solution trajectories exhibit a different behaviour. Consider the SSG trajectory in figure 1, the initial condition of which (marked by $\square$) is less anisotropic (in magnitude of $b_{ij}b_{ij}$) than the equilibrium point. While the Reynolds stress appears realizable at all times along the trajectory, pressure–strain correlation realizability violation (of equation (4.16)) occurs at the point marked by $\star$ close to the two-componential limit of Reynolds stress. For this trajectory, the pressure-gradient variance also becomes negative. This behaviour is entirely inconsistent with Navier–Stokes equations and must be considered physically impossible. Yet, such behaviour would be permitted by the strong realizability constraint. This is a clear indication of the inadequacy of even the strong realizability approach.

Plane strain. Some sample trajectories calculated in the plane-strain mean flow using the LRR model are shown in figure 2. Unrealizable trajectories are again encountered (as described in the plane-strain case) depending upon the initial level of anisotropy. Yet again, the pressure–strain correlation realizability violation (marked with $\star$) precedes that of Reynolds stress (denoted by $\times$). It should be noted that initial

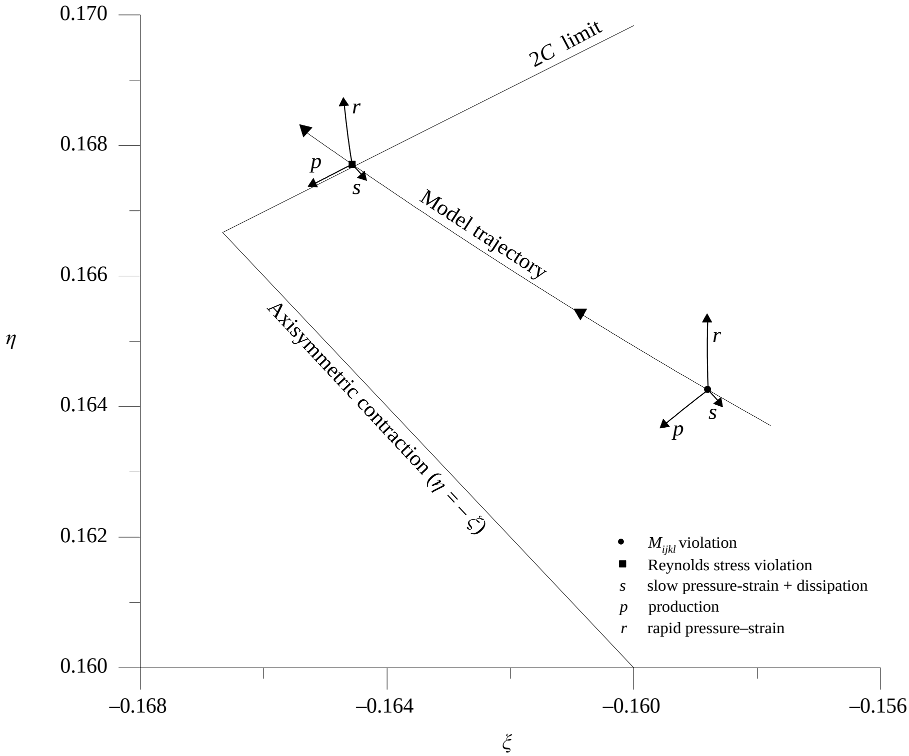

FIGURE 3. Contributions of individual components at different stages of anisotropy evolution in plain-strain flow computation using LRR model ($SK_0/\varepsilon_0 = 5$).

conditions only marginally more anisotropic than the equilibrium point lead to unrealizable trajectories.

Cause of violation

To better comprehend the sequence of model events that lead to unrealizable Reynolds stress, we now study the contribution of various terms in the transport equation – production, rapid pressure–strain correlation, slow pressure–strain correlation and dissipation – towards Reynolds stress evolution. In figure 3, we show an unrealizable LRR model trajectory in the plane-strain case ($Sk_0/\varepsilon_0 = 5$). Thus, the case considered is that of weakly strained flow, typical of situations encountered in many engineering applications. Along the trajectory, the contribution of various terms during two stages of evolution are marked with arrows. The two stages considered are (i) the location of the first pressure–strain correlation constraint violation (marked in the figure with a filled circle); and (ii) the two-componential state where the Reynolds stress realizability violation occurs (marked with a filled square). Arrows indicate the direction of evolution due to individual terms. The length of the arrow represents the relative magnitude of that term. The overall evolution is along the direction of the vector sum of the various arrows. The contributions of slow pressure–strain correlation and dissipation are combined into one as they reflect the nonlinear processes. The roles of various terms are evident from the figure. Production (indicated by the arrow marked p) causes the Reynolds stress to evolve in a direction parallel to the two-component line of the Lumley triangle. It is clear that production cannot cause the violation. The nonlinear term (arrow s) is much smaller in magnitude than the other two. It leads to evolution that returns the Reynolds stress towards isotropy and, hence, cannot be the cause of the violation. The rapid pressure–strain correlation term (arrow r) is of comparable magnitude to production and points in a direction directly out of the Lumley triangle. This term, unambiguously, is the reason for Reynolds stress realizability violation. The overall implication from the figure is that Reynolds stress realizability violation is triggered by the unphysical rapid pressure–strain correlation model. Several other unrealizable trajectories were also studied (Sambasivam *et al.* 2004) and the conclusions are the same: (i) rapid pressure–strain correlation causes Reynolds stress realizability violation; (ii) the new constraint recognizes the deficiency in the rapid pressure–strain correlation long before the trajectory reaches the two-component limit.

Nonlinear pressure–strain correlation models

We now investigate whether nonlinear rapid pressure–strain correlation models, that satisfy the strong realizability constraints, violate the constraints derived in this paper. Three models are considered: the Shih–Lumley model (Shih & Lumley 1985) and two versions of closure expressions given in Sjögren & Johansson (2000). Calculations reveal that though these models may produce Reynolds stresses that are always realizable, they may or may not satisfy the new constraints. The region of initial turbulence states in the Lumley triangle in which the Shih–Lumley model violates the rapid pressure–strain correlation is shown in figure 4 for axisymmetric contraction (filled circle) and plane strain (filled square) flows. Initial conditions above the marked line can yield a trajectory with unrealizable rapid pressure–strain correlation. The first of the two models of Sjögren & Johansson (2000) tested – given in equations 75 and 76 – was of order three with rapid pressure–strain model coefficients: $\gamma_1 = -0.05$, $\gamma_2 = -(3/88) + (21/22)\gamma_1$, $\gamma_3 = (3/88) - (391/110)\gamma_1$ and $\gamma_4 = 0$. It was found that this model yielded largely unrealizable results. It was brought to our notice that these coefficients were misprinted in the published article (A. V. Johansson, personal communication). The second model tested (developed in Johansson & Hallback 1994) was calibrated in a rapidly distorted homogeneous flow to ensure the correct response to various forms of sudden strain and shear. The model (given in equations 96 and 97 of Sjögren & Johansson 2000) is of order four with coefficients: $\gamma_1 = -0.143$, $\gamma_2 = 0.0295$, $\gamma_3 = -0.0484$ and $\gamma_4 = 14.0$. The results of this model yielded fully realizable Reynolds stress and also satisfied the new constraints over most of the Lumley triangle. The new constraints were violated only over a very small region near the two-component line. Sjögren & Johansson (2000) suggest a different fourth-order model which is calibrated to perform better in wall-bounded flows. This model was not tested, since homogeneous flows are the focus of this study.

Overall, it is clear from the failure of the Shih–Lumley model and the unrealizable trajectory (starting from □) in figure 1 that the strong realizability alone cannot guarantee consistency with the new realizability requirements proposed in this paper.

5.1. *Utility of new constraints*

We will now recast constraint 3 in a form that will lead to constraints on the coefficients in the rapid pressure–strain correlation closure expression (5.1). Since the Reynolds stress tensor is symmetric, there exists a coordinate system in which it can be diagonalized. We will consider constraint 3 and (2.9) in this coordinate system.

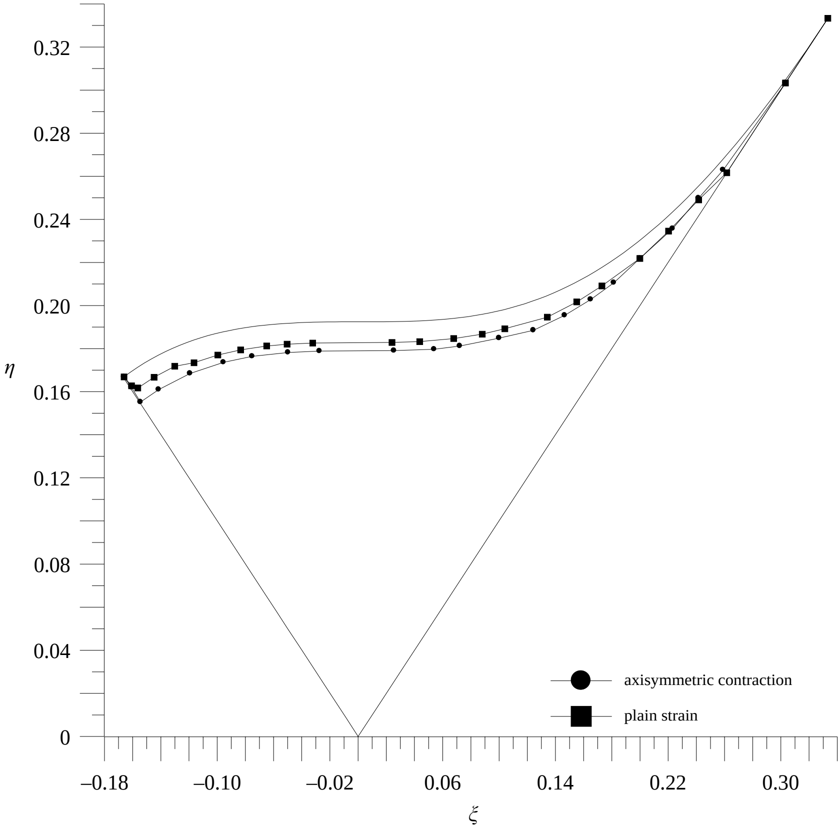

FIGURE 4. Rapid pressure–strain correlation unrealizable-region map of the Shih–Lumley (1985) model.

The constraint can be recast as

$$\left| \frac{\partial \overline{U_k}}{\partial x_l} M_{\alpha l \alpha k} \right| \leqslant (2\overline{u_\alpha u_\alpha})^{1/2} \left[\frac{\partial \overline{U_j}}{\partial x_i} \frac{\partial \overline{U_k}}{\partial x_l} M_{iljk} \right]^{1/2}. \tag{5.6}$$

This inequality can be further rewritten as

$$\frac{1}{\sqrt{2}} \left| \frac{\partial \overline{U_q}}{\partial x_p} M_{\alpha q \alpha p} \right| \Big/ \left[\frac{\partial \overline{U_j}}{\partial x_i} \frac{\partial \overline{U_k}}{\partial x_l} M_{iljk} \right]^{1/2} \leqslant \overline{u_\alpha u_\alpha}^{1/2}. \tag{5.7}$$

In the two-componential limit, when $\overline{u_\alpha u_\alpha}$ is zero, the above inequality can be recognized as the strong realizability constraint. In this light, the new constraint can be thought of as a more general realizability constraint that restricts rapid pressure–strain correlation at all values of Reynolds stress. Unlike the strong and weak realizability constraints, the new requirements involve the mean flow gradients, and hence are flow specific. Mean-flow specific constraints are generally more useful for model suggestion and calibration. These constraints could perhaps lead to rapid pressure–strain correlation models that satisfy realizability, without compromising on the other requirements. Using these constraints to develop models from first principles will be deferred to future work.

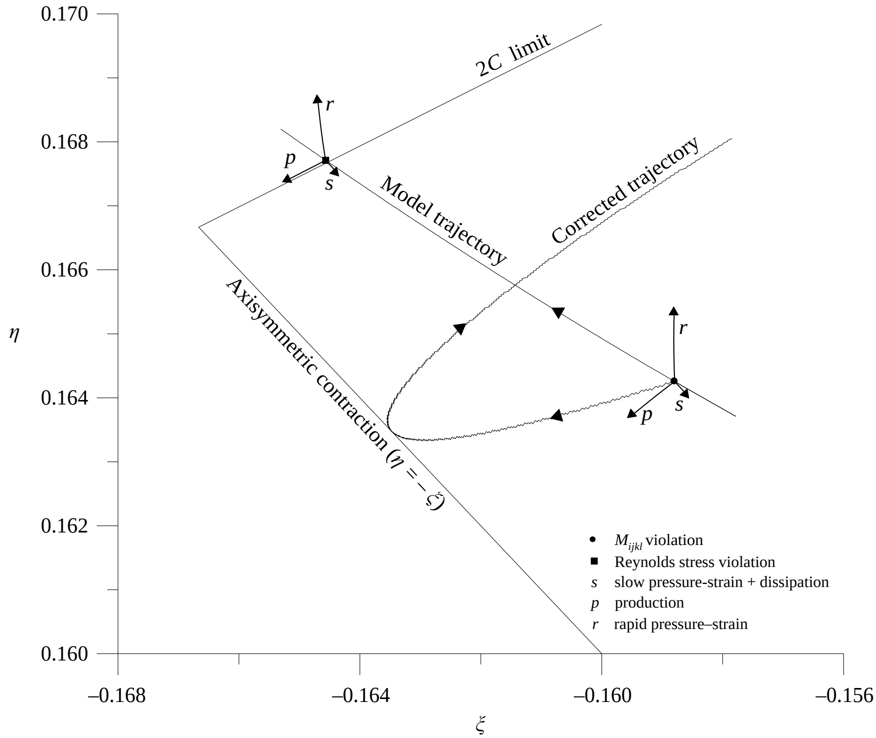

FIGURE 5. Contributions of individual components at different stages of anisotropy evolution in plain-strain flow computation using truncated LRR model ($SK_0/\varepsilon_0 = 5$).

Fully realizable truncated LRR model

Here, we will demonstrate the potential promise of the current constraints by simply truncating the LRR model to yield a fully realizable model. The LRR model is chosen for this exercise as it satisfies all of the required constraints except realizability. To achieve a fully realizable trajectory, the model rapid pressure–strain correlation should be corrected when it violates the new constraints. The rapid pressure–strain correlation is set to zero when it violates constraints 2 or 3:

$$\phi_{ij}^{'t} = \begin{cases} \phi_{ij}' & \text{when constraints are not violated,} \\ 0 & \text{when constraints are violated.} \end{cases} \tag{5.8}$$

Calculations using this truncated model reveal no Reynolds stress realizability violation for any of the initial conditions in all mean flows considered. A sample evolution trajectory computed with this model is shown in figure 5. The flow is the case considered in figure 3 where it was computed with the original LRR model. The two trajectories are identical until the original LRR model violates the rapid pressure–strain realizability constraint. After this stage, the truncated model evolution is quite different. The trajectory becomes nearly parallel to the two-component line and approaches the axisymmetric contraction line. Ultimately, it asymptotes the same equilibrium point as that of all realizable trajectories. Several corrected trajectories in various types of mean flow are shown in Sambasivam *et al.* (2004). Thus, truncating the LRR model with the new constraint yields a fully realizable and piecewise linear rapid pressure–strain correlation model.

We now address the questions raised at the end of the last section.

(i) Do the current models violate the new realizability constraints? Yes, the current models do violate the pressure–strain correlation constraints. In fact, this transgression is more rampant than Reynolds-stress realizability violation.

(ii) Does Reynolds-stress realizability automatically imply realizability of pressure–strain correlation? No. Even models satisfying the strong realizability requirement violate the new constraints.

(iii) Does pressure–strain correlation realizability guarantee Reynolds-stress realizability? The strong realizability requirement (which guarantees realizable Reynolds stress) is a special limit of the more general constraint 3. Hence, rapid pressure–strain correlation realizability will guarantee that Reynolds stress will not become unrealizable owing to this model term. An example of this can be seen in figure 5.

(iv) Are old and new realizability constraints mutually exclusive so that both should be required of the pressure–strain correlation models? As already pointed out, the strong realizability constraint is an extreme limit of the constraint derived in this paper. Therefore, the pressure–strain correlation constraint is more fundamental than the Reynolds stress realizability constraint.

6. Summary and conclusion

We propose an extended realizability condition that requires that all the statistical moments contributing towards Reynolds stress evolution be individually realizable. Such a requirement can lead to realizable Reynolds stress and realizable contributing moments, unlike current approaches. This will ensure a higher degree of fidelity to the governing Navier–Stokes equations than those satisfying only the Schumann constraints.

Towards accomplishing the above goal, two new constraints on the rapid velocity pressure-gradient correlation are derived. The constraints are: (i) pressure-gradient variance must be non-negative leading to the requirement that the M_{ijkl} tensor is positive semi-definite; and, (ii) the pressure–strain correlation model expression must satisfy the Schwarz inequality. It is shown that the strong realizability requirement is a special case (in the two-componential limiting state of turbulence) of the more general constraints developed in this paper.

A thorough computational investigation of realizability violations in currently popular pressure–strain correlation models is performed. The important findings from the calculations and analysis are

(i) Strong or weak realizability constraints cannot guarantee realizable pressure–strain correlation.

(ii) Every observed episode of unrealizable Reynolds stress in the model calculation is preceded by an unrealizable pressure–strain correlation.

(iii) An unphysical contributing closure model is a root cause of Reynolds stress realizability violation.

(iv) A fully realizable piecewise linear pressure–strain correlation model can be obtained by truncating the LRR model as dictated by the new constraints.

Our next step is to develop new rapid pressure–strain correlation models from first principles employing the new constraints. Further, realizability of the slow pressure–strain correlation model will be carefully studied. We will also explore the possibility of developing more stringent constraints on M_{ijkl} using the turbulence structure tensors of Kassinos *et al.* (2001).

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